

VANISHING OF ℓ^2 -BETTI NUMBERS OF LOCALLY COMPACT GROUPS AS AN INVARIANT OF COARSE EQUIVALENCE

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ABSTRACT. We provide a short proof that the vanishing of ℓ^2 -Betti numbers of unimodular locally compact second countable groups is an invariant of coarse equivalence.

1. INTRODUCTION

The insight that the vanishing of ℓ^2 -Betti numbers provides a quasi-isometry invariant is due to Gromov [11, Chapter 8], and positive results around this insight have a long history. The most important contribution is by Pansu [17] whose work on asymptotic ℓ^p -cohomology includes the fact that the vanishing of ℓ^2 -Betti numbers of discrete groups of type F_∞ , is a quasi-isometry invariant.

There is a growing interest in the metric geometry of locally compact groups [2, 3]. We thus think it is important to have the quasi-isometry and coarse invariance of the vanishing of ℓ^2 -Betti numbers available in the greatest generality. Following Pansu's ideas and relying on more recent advances in the theory of ℓ^2 -Betti numbers, we provide a proof of the following result.

Theorem 1. *Let G and H be unimodular locally compact second countable groups. If G and H are coarsely equivalent then the n -th ℓ^2 -Betti number of G vanishes if and only the n -th ℓ^2 -Betti number of H vanishes.*

The coarse invariance for discrete groups was proven earlier in a paper of Mimura-Ozawa-Sako-Suzuki [15, Corollary 6.3].

Every locally compact, second countable group G (hereafter abbreviated by **lcsc**) has a left-invariant proper continuous metric by a theorem of Struble [25]. As any two left-invariant proper continuous metrics on G are coarsely equivalent, every lcsc group has a well defined coarse geometry. Further, any coarse equivalence between compactly generated lcsc groups is a quasi-isometry with respect to word metrics of compact symmetric generating sets and vice versa. See [3, Chapter 4] for a systematic discussion of these notions.

To even state Theorem 1 in that generality, recent advances in the theory of ℓ^2 -Betti numbers were necessary. ℓ^2 -Betti numbers of discrete groups enjoy a long history but it was not until recently that ℓ^2 -Betti numbers were defined for arbitrary unimodular lcsc groups by Petersen [18], and a systematic theory analogous to the discrete case emerged [12, 18, 19]. Earlier studies of ℓ^2 -Betti numbers of locally compact groups in specific cases can be found in [4, 5, 9].

2010 *Mathematics Subject Classification.* Primary 20F65; Secondary 22D99.

Key words and phrases. Coarse geometry, locally compact groups, ℓ^2 -Betti numbers.

We acknowledge support by the German Science Foundation via the Research Training Group 2229.

Previous results on coarse invariance. Pansu [17] introduced asymptotic ℓ^p -cohomology and proved its invariance under quasi-isometries. If a group Γ is of type F_∞ , then the ℓ^p -cohomology of Γ coincides with its asymptotic ℓ^p -cohomology [17, Théorème 1]. The geometric explanation for the appearance of the type F_∞ condition is that the finite-dimensional skeleta of the universal covering of a classifying space of finite type are uniformly contractible. As an immediate consequence of Pansu's result, the vanishing of ℓ^2 -Betti numbers is a quasi-isometry invariant among discrete groups of type F_∞ . The same arguments work for totally disconnected groups admitting a topological model of finite type [22].

Elek [6] investigated the relation between ℓ^p -cohomology of discrete groups and Roe's coarse cohomology and proved similar results. Another independent treatment is due to Fan [7]. Genton [10] elaborated upon Pansu's methods in the case of metric measure spaces.

Oguni [16] generalised the quasi-isometry invariance of the vanishing of ℓ^2 -Betti numbers from discrete groups of type F_∞ to discrete groups whose cohomology with coefficients in the group von Neumann algebra satisfies a certain technical condition. A similar technical condition appears in the proof of quasi-isometry invariance of Novikov-Shubin invariants of amenable groups [24], and it is unclear how much this condition differs from the type F_∞ -condition. Oguni's groupoid approach is inspired by [8, 24] and quite different from the approaches by Elek, Fan, and Pansu.

The coarse invariance of vanishing of ℓ^2 -Betti numbers for discrete groups was shown by Mimura-Ozawa-Sako-Suzuki [15, Corollary 6.3]. Li [13] recently reproved this using groupoid techniques as a consequence of more general cohomological coarse invariance results.

Structure of the paper. We review the necessary basics of ℓ^2 -Betti numbers and continuous cohomology in Section 2. In Section 3 we define coarse ℓ^2 -cohomology for lcsc groups and show that it is isomorphic to continuous cohomology. In Section 4 we conclude the proof of Theorem 1.

2. CONTINUOUS COHOMOLOGY AND ℓ^2 -BETTI NUMBERS OF LCSC GROUPS

Let G be a unimodular lcsc group with Haar measure μ . Let X be a locally compact second countable space with Radon measure ν . Let E be a Fréchet space.

The space $C(X, E)$ of continuous functions from X to E becomes a Fréchet space when endowed with the topology of compact convergence. Let $L^2_{loc}(X, E)$ be the space of equivalence classes of measurable maps $f: X \rightarrow E$ up to ν -null sets such that $\|f|_K\|_E$ is square-integrable for every compact subset $K \subset X$. The L^2 -norm of the function $\|f|_K\|_E$ defines a semi-norm p_K on $L^2_{loc}(X, E)$. The family of semi-norms p_K , $K \subset E$, turns $L^2_{loc}(X, E)$ into a Fréchet space.

We call a Fréchet space E with a continuous (i.e. $G \rightarrow E$, $g \mapsto gv$, is continuous for every $v \in E$) linear G -action a G -module. A continuous linear G -equivariant map between G -modules is a *homomorphism of G -modules*. If E is a G -module and G acts continuously and ν -preserving on X then $C(X, E)$ and $L^2_{loc}(X, E)$ become G -modules via $(g \cdot f)(x) = gf(g^{-1}x)$ for $x \in X$ and $g \in G$ [1, Proposition 3.1.1]. The usual homogeneous coboundary map

$$(1) \quad d^{n-1}f(g_0, \dots, g_n) = \sum_{i=0}^n (-1)^i f(g_0, \dots, \widehat{g_i}, \dots, g_n)$$

defines cochain complexes $C(G^{*+1}, E)$ and $L_{loc}^2(G^{*+1}, E)$ of G -modules (cf. [1, Proposition 3.2.1]). Here we take the diagonal G -action on G^{*+1} . We recall the following definition.

Definition 2. The *(continuous) cohomology* of G in E is the cohomology

$$H^n(G, E) = H^n(C(G^{*+1}, E)^G)$$

of the G -invariants of $C(G^{*+1}, E)$. The *reduced (continuous) cohomology* $\underline{H}^*(G, E)$ is a quotient of $H^*(G, E)$ obtained by taking the quotient with the closure of $\text{im } d^{*-1}$ instead of $\text{im } d^{*-1}$.

We have an obvious inclusion

$$(2) \quad I^* : C(G^{*+1}, E) \rightarrow L_{loc}^2(G^{*+1}, E).$$

The maps I^* form a cochain map of G -modules. Taking a positive function $\chi \in C_c(G)$ there is a cochain map $R^* : L_{loc}^2(G^{*+1}, E) \rightarrow C(G^{*+1}, E)$ of G -modules

$$(R^n f)(g_0, \dots, g_n) = \int_{G^{n+1}} f(h_0, \dots, h_n) \chi(g_0^{-1} h_0) \cdot \dots \cdot \chi(g_n^{-1} h_n) d\mu(h_0, \dots, h_n)$$

such that $I^* \circ R^*$ and $R^* \circ I^*$ are homotopic (as cochain maps of G -modules) to the identity [1, Proposition 4.8]. So we have the following useful fact:

Theorem 3. *The cochain map I^* in (2) induces isomorphisms in cohomology and in reduced cohomology.*

Next we turn to the case where the coefficient module $E = L^2(G)$ is the regular representation, relevant for the definition of ℓ^2 -Betti numbers.

Let $L(G)$ be the *von Neumann algebra* of G ; the Haar measure μ defines a semifinite trace tr_μ on $L(G)$. There are a natural left G -action and a natural right $L(G)$ -action on $L^2(G)$, and the two actions commute. Hence also the G -actions on $C(G^{*+1}, L^2(G))$ and $L_{loc}^2(G^{*+1}, L^2(G))$ considered previously and the $L(G)$ -actions induced from the right $L(G)$ -action on $L^2(G)$ commute. So the (reduced and non-reduced) continuous cohomology of G in $L^2(G)$ is naturally a $L(G)$ -module¹. Obviously, the cochain map I^* above is compatible with the $L(G)$ -module structures. The groups $H^*(G, L^2(G))$ are called the *(continuous) ℓ^2 -cohomology* of G . Similarly for the reduced cohomology.

Petersen [18] extended Lück's dimension function from finite von Neumann algebras to semifinite von Neumann algebras. The dimension function \dim_μ with respect to (G, μ) is a non-trivial dimension for (algebraic) right $L(G)$ -modules that is additive for short exact sequences of $L(G)$ -modules. It scales as $\dim_{c\mu} = c^{-1} \dim_\mu$ for $c > 0$. The fact that a $L(G)$ -module has dimension zero can be expressed without referring to the trace: it is an algebraic fact. The following criterion was shown by the first author for finite von Neumann algebras [23, Theorem 2.4]; it was extended to the semifinite case by Petersen [18, Lemma B.27].

Theorem 4. *An $L(G)$ -module M satisfies $\dim_\mu(M) = 0$ if and only if for every $x \in M$ there is an increasing sequence (p_i) of projections in $L(G)$ with $\sup p_i = 1$ such that $x p_i = 0$ for every $i \in \mathbb{N}$.*

¹When talking about $L(G)$ -modules we mean the algebraic module structure and ignore topologies.

Definition 5. The n -th ℓ^2 -Betti number of G is the $L(G)$ -dimension of its reduced continuous cohomology with coefficients in $L^2(G)$, i.e.

$$\beta_{(2)}^n(G) := \dim_{\mu} \underline{H}^n(G, L^2(G)) \in [0, \infty].$$

Remark 6. Equivalently, the n -th ℓ^2 -Betti number can be defined as the $L(G)$ -dimension of the non-reduced cohomology $H^n(G, L^2(G))$. This is a non-trivial fact (see [12, Theorem A]). For discrete G , our definition coincides with Lück's definition in [14]. Again, this is non-trivial and shown in [20, Theorem 2.2].

The following lemma was observed in [18, Proposition 3.8]. Since it is a direct consequence of Theorem 4 we present the argument.

Lemma 7. $\beta_{(2)}^n(G) = 0 \Leftrightarrow \underline{H}^n(G, L^2(G)) = 0$.

Proof. Let $\beta_{(2)}^n(G) = 0$. Let $f: G^{n+1} \rightarrow L^2(G)$ be a cocycle representing a cohomology class $[f]$ in $\underline{H}^n(G, L^2(G))$. By Theorem 4 there is an increasing sequence of projections $p_j \in L(G)$ whose supremum is 1 such that each fp_j is a coboundary $d^{n-1}b_j$. It is clear that $fp_j = d^{n-1}b_j$ converges to f in the topology of $C(G^{n+1}, L^2(G))$, thus $[f] = 0$. \square

3. COARSE EQUIVALENCE AND COARSE ℓ^2 -COHOMOLOGY

Let G be a unimodular lcsc group. We fix a left-invariant proper continuous metric d on G . Let μ be a Haar measure on G . Let μ_n be the n -fold product measure of μ on G^n .

For every $R > 0$ and $n \in \mathbb{N}_0$ we consider the closed subset

$$G_R^n := \{(g_0, \dots, g_{n-1}) \in G^n \mid d(g_i, g_j) \leq R \text{ for all } 0 \leq i, j \leq n-1\}$$

and a family of semi-norms for measurable maps $\alpha: G^{n+1} \rightarrow \mathbb{C}$ defined by

$$\|\alpha\|_R^2 = \int_{G_R^{n+1}} |\alpha(g_0, \dots, g_n)|^2 d\mu_{n+1} \in [0, \infty].$$

Let $CX_{(2)}^n(G)$ be the space of equivalence classes (up to μ_{n+1} -null sets) of measurable maps $\alpha: G^{n+1} \rightarrow \mathbb{C}$ such that $\|\alpha\|_R < \infty$ for every $R > 0$. The semi-norms $\|\cdot\|_R$, $R > 0$, turn $CX_{(2)}^n(G)$ into a Fréchet space. It is straightforward to verify that the homogeneous differential (1) yields a well-defined, continuous homomorphism $CX_{(2)}^n(G) \rightarrow CX_{(2)}^{n+1}(G)$ (cf. [10, Proposition 2.3.3]). Thus we obtain a cochain complex of Fréchet spaces.

Definition 8. The *coarse ℓ^2 -cohomology* of G is defined as

$$HX_{(2)}^n(G) = H^n(CX_{(2)}^*(G)).$$

By taking the quotients by the closure of the differentials, one defines similarly the *reduced coarse ℓ^2 -cohomology* $\underline{H}X_{(2)}^n(G)$.

Remark 9. The previous definition is the continuous analog of Elek's definition [6, Definition 1.3] in the discrete case (Elek gives credits to Roe [21]). It is very much related to Pansu's *asymptotic ℓ^2 -cohomology* [17], which was considered in the generality of metric measure spaces by Genton [10]. The difference of our definition to the one in Genton [10] is as follows: $CX_{(2)}^*(G)$ is an inverse limit of spaces $L^2(G_R^{*+1})$. Unlike us, Genton takes first the cohomology of $L^2(G_R^{*+1})$ and

then the inverse limit. Under some uniform contractibility assumptions the two definitions coincide but likely not in general.

Theorem 10. *Let G be a unimodular lcsc group. For every $n \geq 0$, there are isomorphisms of abelian groups*

$$\begin{aligned} H^n(G, L^2(G)) &\cong HX_{(2)}^n(G), \\ \underline{H}^n(G, L^2(G)) &\cong \underline{H}X_{(2)}^n(G). \end{aligned}$$

Proof. We have the obvious embedding

$$L_{loc}^2(G^{n+1}, L^2(G)) \subset L_{loc}^2(G^{n+1}, L_{loc}^2(G))$$

and the exponential law (see [1, Lemme 1.4] for a proof but beware of the typo in the statement)

$$L_{loc}^2(G^{n+1}, L_{loc}^2(G)) \cong L_{loc}^2(G^{n+1} \times G).$$

Thus an element in $L_{loc}^2(G^{n+1}, L^2(G))^G$ is represented by a measurable complex function in $(n+2)$ -variables. For $\alpha \in L_{loc}^2(G^{n+1}, L^2(G))^G$ we define μ_{n+2} -almost everywhere

$$F^n(\alpha)(x_0, \dots, x_n, x) = \alpha(x^{-1}x_0, \dots, x^{-1}x_n)(x).$$

The measurable function $F^n(\alpha)$ is invariant by translation in the $(n+2)$ -th variable. By Fubini's theorem we may regard $F^n(\alpha)$ as a measurable function $E^n(\alpha): G^{n+1} \rightarrow \mathbb{C}$ in the first $(n+1)$ -variables. We may think of $E^n(\alpha)$ as an evaluation of α at e . Let $B(R)$ denote the R -ball around $e \in G$. Next we show that $\|E^n(\alpha)\|_R < \infty$ for every $R > 0$, thus $E^n(\alpha) \in CX_{(2)}^n(G)$.

Since $\alpha \in L_{loc}^2(G^{n+1}, L^2(G))^G$ we have

$$\begin{aligned} \infty &> \int_{B(2R)^{n+1}} \int_G |\alpha(x_0, x_1, \dots, x_n)(x)|^2 d\mu d\mu_{n+1} \\ &= \int_{B(2R)^{n+1}} \int_G |\alpha(x, xx_0^{-1}x_1, \dots, xx_0^{-1}x_n)(x_0)|^2 d\mu d\mu_{n+1}. \end{aligned}$$

The map

$$m: G^{n+2} \rightarrow G^{n+2}, (x_0, \dots, x_n, x) \mapsto (x, xx_0^{-1}x_1, \dots, xx_0^{-1}x_n, x_0)$$

is measure preserving. Further, we have

$$m(G_R^{n+1} \times B(R)) \subset B(2R)^{n+1} \times G.$$

This implies the first inequality below. The first equality follows from the fact that $(x_0, \dots, x_n, x) \mapsto (x^{-1}x_0, \dots, x^{-1}x_n, x)$ is a measure preserving measurable automorphism of $G_R^{n+1} \times B(R)$.

$$\begin{aligned} \int_{B(2R)^{n+1}} \int_G |\alpha(x, xx_0^{-1}x_1, \dots, xx_0^{-1}x_n)(x_0)|^2 d\mu d\mu_{n+1} \\ \geq \int_{G_R^{n+1}} \int_{B(R)} |\alpha(x_0, \dots, x_n)(x)|^2 d\mu d\mu_{n+1} \\ = \int_{G_R^{n+1}} \int_{B(R)} |\alpha(x^{-1}x_0, \dots, x^{-1}x_n)(x)|^2 d\mu d\mu_{n+1} \\ = \mu(B(R)) \|E^n(\alpha)\|_R. \end{aligned}$$

Hence $\|E^n(\alpha)\|_R$ is finite for every $R > 0$. That $E^*: L^2_{loc}(G^{*+1}, L^2(G))^G \rightarrow CX^*_{(2)}(G)$ defines a cochain map is obvious. The above computation also implies that E^* is continuous with respect to the Fréchet topologies.

Given $\beta \in CX^n_{(2)}(G)$ we define

$$M^n(\beta)(g_0, \dots, g_n)(g) = \beta(g^{-1}g_0, \dots, g^{-1}g_n)$$

for μ_{n+2} -almost every (g_0, \dots, g_n, g) . The function $M^n(\beta)$ defines an element in $L^2_{loc}(G^{n+1}, L^2(G))^G$. The G -invariance of $M^n(\beta)$ is obvious. We have to show that $\|M^n(\beta)\|_{B(R)^{n+1}}$ is square-integrable for every $R > 0$. This follows from the following computations which is based on the arguments above in reversed order.

$$\begin{aligned} \mu(B(R)) \int_{G^{n+1}_{2R}} |\beta(g_0, \dots, g_n)|^2 d\mu_{n+1} &= \int_{G^{n+1}_{2R}} \int_{B(R)} |\beta(g_0, \dots, g_n)|^2 d\mu d\mu_{n+1} \\ &\geq \int_{B(R)^{n+1}} \int_G |\beta(g^{-1}g_0, \dots, g^{-1}g_n)| d\mu d\mu_{n+1} \end{aligned}$$

Obviously, M^* is a chain map. Continuity follows from the previous computation. It is clear that M^* and E^* are mutual inverses. Using Theorem 3, this concludes the proof. \square

4. COARSE INVARIANCE

We recall the notion of coarse equivalence. A map $f: (X, d_X) \rightarrow (Y, d_Y)$ between metric spaces is *coarse Lipschitz* if there is a non-decreasing function $a: [0, \infty) \rightarrow [0, \infty)$ with $\lim_{t \rightarrow \infty} a(t) = \infty$ such that

$$d_Y(f(x), f(x')) \leq a(d_X(x, x'))$$

for all $x, x' \in X$. We say that two such maps f, g are *close* if

$$\sup_{x \in X} d_Y(f(x), g(x)) < \infty.$$

A coarse Lipschitz map $f: X \rightarrow Y$ is a *coarse equivalence* if there is a coarse Lipschitz map $g: Y \rightarrow X$ such that fg and gf are close to the identity. We say g is a *coarse inverse* of f .

Lemma 11. Coarsely equivalent lcsc groups are measurably coarse equivalent, i.e. if G and H are coarse equivalent lcsc groups then there are *measurable* coarse Lipschitz maps $f: G \rightarrow H$ and $g: H \rightarrow G$ such that fg and gf are close to the identity.

Proof. We choose left-invariant continuous proper metrics d_G and d_H on G and H , respectively. Let $f: G \rightarrow H$ be a coarse Lipschitz map with $d_H(f(x), f(x')) \leq a(d_G(x, x'))$. Let $t > 0$. We pick a countable measurable partition \mathcal{U} of G whose elements have diameter $\leq t$ and choose an element $x_U \in U$ for every $U \in \mathcal{U}$. By setting $\tilde{f}(x) = f(x_U)$ for $x \in U$ we obtain a coarse Lipschitz map $\tilde{f}: G \rightarrow H$ which satisfies $d(\tilde{f}(x), \tilde{f}(x')) \leq a(d(x, x')) + 2t$ and is close to f with $d(\tilde{f}(x), f(x)) \leq a(2t)$. Analogously, we construct a measurable coarse Lipschitz map \tilde{g} , constructed from a coarse Lipschitz map $g: H \rightarrow G$ which is a coarse inverse to f . It is obvious that \tilde{g} is a coarse inverse to \tilde{f} . \square

Theorem 12. *Coarsely equivalent unimodular lcsc groups have isomorphic reduced and non-reduced coarse ℓ^2 -cohomology groups.*

Proof. Let G and H lcsc groups with Haar measures μ and ν , respectively. Let $f: G \rightarrow H$ be a coarse equivalence with coarse inverse g . Because of lemma 11 we can further assume that f and g are measurable. We define a map $\chi: G \times G \rightarrow \mathbb{R}$ by

$$\chi(x, y) = \frac{\mathbb{1}_{B_x(c)}(y)}{\mu(B(c))}$$

where we choose c such that $\mu(B(c)) \geq 1$. Then χ is a measurable function with $\chi(x, y) = \chi(y, x)$ and $\int_G \chi(x, y) d\mu(y) = 1$. We use the following notation:

$$\chi: G^{n+1} \times G^{n+1} \rightarrow \mathbb{R}, \quad \chi((x_0, \dots, x_n), (y_0, \dots, y_n)) = \chi(x_0, y_0) \cdot \dots \cdot \chi(x_n, y_n).$$

Analogously, we define $\chi': H^{n+1} \times H^{n+1} \rightarrow \mathbb{R}$ with some radius c' . Now we can define the maps $f^*: HX_{(2)}^*(H) \rightarrow HX_{(2)}^*(G)$ and $g^*: HX_{(2)}^*(G) \rightarrow HX_{(2)}^*(H)$ as follows where we use x_i for elements in G and y_i for elements of H :

$$\begin{aligned} f^* \alpha(x_0, \dots, x_n) &= \int_{H^{n+1}} \alpha(y_0, \dots, y_n) \chi'((f(x_0), \dots, f(x_n)), (y_0, \dots, y_n)) d\nu_{n+1} \\ g^* \beta(y_0, \dots, y_n) &= \int_{G^{n+1}} \beta(x_0, \dots, x_n) \chi((g(y_0), \dots, g(y_n)), (x_0, \dots, x_n)) d\mu_{n+1}. \end{aligned}$$

The idea of averaging over a function like χ goes back to Pansu; it is necessary in our context since the maps f and g do not preserve the measure classes, in general.

First of all, we check that these are well-defined continuous cochain maps.

$$\begin{aligned} \infty &> \|\alpha\|_{a(R)+c'}^2 = \int_{H^{n+1}} |\alpha(y_0, \dots, y_n)|^2 \cdot \mathbb{1}_{H_{a(R)+c'}^n} d\nu_{n+1} \\ &\geq \int_{H^{n+1}} |\alpha(y_0, \dots, y_n)|^2 \int_{G_R^{n+1}} \chi'((f(x_0), \dots, f(x_n)), (y_0, \dots, y_n)) d\mu_{n+1} d\nu_{n+1} \\ &= \int_{G_R^{n+1}} \int_{H^{n+1}} |\alpha(y_0, \dots, y_n)|^2 \chi'((f(x_0), \dots, f(x_n)), (y_0, \dots, y_n)) d\nu_{n+1} d\mu_{n+1} \\ &\geq \int_{G_R^{n+1}} \left| \int_{H^{n+1}} \alpha(y_0, \dots, y_n) \chi'((f(x_0), \dots, f(x_n)), (y_0, \dots, y_n)) d\nu_{n+1} \right|^2 d\mu_{n+1} \\ &= \int_{G_R^{n+1}} |f^n \alpha(x_0, \dots, x_n)|^2 d\mu_{n+1} = \|f^n \alpha\|_R^2 \end{aligned}$$

It is a direct computation that $d^n \circ f^n = f^{n+1} \circ d^n$. It remains to show that there is a cochain homotopy $h: CX_{(2)}^*(H) \rightarrow CX_{(2)}^{*-1}(H)$ such that $\text{Id} - g^* f^* = hd + dh$. We define $h_i^{n+1}: CX_{(2)}^{n+1}(H) \rightarrow CX_{(2)}^n(H)$ by

$$h_i^{n+1} \alpha(y_0, \dots, y_n) = \int_{H^{n+1}} \alpha(\tilde{y}_0, \dots, \tilde{y}_i, y_i, \dots, y_n) \chi'((y_0, \dots, y_n), (\tilde{y}_0, \dots, \tilde{y}_n)) d\nu_{n+1}(\tilde{y})$$

and set

$$h^{n+1} = \sum_{i=0}^n (-1)^i h_i^{n+1}.$$

That h^* is well-defined is a similar consideration as to show that f^* and g^* are well-defined. Now let us denote the i -th term of the coboundary map by d_i^n , i.e.

$d_i^n \alpha(y_0, \dots, y_{n+1}) = \alpha(y_0, \dots, \widehat{y}_i, \dots, y_{n+1})$. It is straightforward to verify that we have the following relations:

$$\begin{aligned} h_n^{n+1} \circ d_{n+1}^n &= g^n \circ f^n, \\ h_0^{n+1} \circ d_0^n &= \text{Id}_{CX_{(2)}^n(H)}, \\ h_j^{n+1} \circ d_i^n &= d_i^{n-1} \circ h_{j-1}^n && \text{for } 1 \leq j \leq n \text{ and } i \leq j, \\ h_j^{n+1} \circ d_i^n &= d_{i-1}^{n-1} \circ h_j^n && \text{for } 1 \leq i \leq n \text{ and } i > j. \end{aligned}$$

We get $h^{n+1}d^n + d^{n-1}h^n = \text{Id}_{CX_{(2)}^n(H)} - g^n \circ f^n$. The same construction applies to f^*g^* which completes the proof. \square

Proof of Theorem 1. Let G and H be unimodular lcsc groups. Let G and H be coarsely equivalent. Then we have the following equivalences:

$$\begin{aligned} \beta_{(2)}^2(G) = 0 &\Leftrightarrow \underline{H}^n(G, L^2(G)) = 0 && \text{(Lemma 7)} \\ &\Leftrightarrow \underline{H}X_{(2)}^n(G) = 0 && \text{(Theorem 10)} \\ &\Leftrightarrow \underline{H}X_{(2)}^n(H) = 0 && \text{(Theorem 12)} \end{aligned}$$

Going the same steps backwards for the group H finishes the proof. \square

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